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ON THE ISOMETRIES OF FOLIATED MANIFOLD*

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Abstract. The purpose of our paper is to introduce a topology on the group $G_F^r(M)$ of all isometries of foliated manifold (M, F) of the class C^r which depends on the foliation F and coincides with compact open topology when F is an n- dimensional foliation. If codimension of F is equal to n, convergence in the introduced topology coincides with pointwise convergence. Some properties of introduced topology are showed.

Keywords: Riemannian manifold, foliation, isometric mapping of foliated manifold, geodesic line, foliated compact open topology.

AMS Subject Classification: Primary 53C12, Secondary 53C22.

1. Introduction

Papers [7], [8] are devoted to isometric mappings of foliated manifold. In these papers it is investigated the question under what conditions any isometry of the foliation will be an isometry of the manifold. In addition it is proved that the existence of a diffeomorfism of a foliated manifold onto itself which is an isometry of the foliation, but it is not an isometry of the manifold. It is constructed an example of a diffeomorfism of three - dimensional sphere which is an isometry of the Hopf fibration but is not an isometry of the three - dimensional sphere.

The purpose of our paper is to study the group $G_F^r(M)$ of isometries of foliated manifold (M, F) with a certain topology, which was introduced in the paper [6], depending on the foliation F, such that it coincides with the compact open topology if F is an n- dimensional foliation. If the codimension of the foliation F is equal to n, convergence in the introduced topology coincides with the pointwise convergence.

2. Preliminaries

Let (M, F_1) and (N, F_2) be n - dimensional smooth foliated manifolds with k dimensional foliations, where $0 < k < n$.

If for some C^r- diffeomorphism $f : M \to N$ the image $f(L_\alpha)$ of any leaf L_α of the foliation F_1 is a leaf of the foliation F_2, we say that pairs (M, F_1) and (N, F_2) C^r- diffeomorphic foliated manifolds. In this case the mapping f is called C^r- diffeomorphism, preserving foliation and is written as

$$f : (M, F_1) \to (N, F_2).$$

In the case where $M = N$, $F_1 = F_2$, f is called a diffeomorphism of the foliated manifold (M, F).

Diffeomorphisms, preserving foliation, are investigated in [1],[9].

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Definition 2.1. *Diffeomorfism $\varphi : M \to M$ of the class $C^r(r \geq 0)$, preserving foliation, is called a foliation isometry F (an isometry of the foliated manifold (M, F)) if it is an isometry on each leaf of the foliation F, i.e. for each leaf L_α of the foliation F, $\varphi : L_\alpha \to \varphi(L_\alpha)$ is an isometry.*

Let M be a n- dimensional smooth connected Riemannian manifold with a Riemannian metric g, F a smooth $k-$ dimensional foliation on M (In this paper manifolds and foliations have C^∞ - smoothness). Let $L(p)$ denote the leaf of the foliation F passing through the point p, by T_pF- tangent space to the leaf $L(p)$ at p, and by H_pF- it's orthogonal complement of T_pF in $T_pM, p \in M$. We get two subbundles (smooth distributions) $TF = \{T_pF : p \in M\}$, $\quad HF = \{H_pF : p \in M\}$ of the tangent bundle TM of the manifold M and, as a result, the tangent bundle TM of the manifold M decomposing into the sum of two orthogonal bundles, i.e. $TM = TF \oplus HF$. The restriction of the Riemannian metric g on T_pF for all p induces a Riemannian metric on the leaves. The induced Riemannian metric a defines distance function on every leaf. Further, throughout in this paper, under the distance on a leaf is understood this distance. This distance on a leaf is different from distance induced by the distance on M.

Let us denote by $G^r_F(M)$ the set of all C^r- isometries of a foliated manifold (M, F), where $r \geq 0$. The following remarks show that the notion of an isometry of a foliated manifold is correctly defined.

Remark 2.1. *If $r \geq 1$, for each element $\varphi \in G^r_F(M)$ the differential $d\varphi$ preserves the length of each tangent vector $\nu \in T_pF$, i.e. $\mid d\varphi_p(\nu) \mid = \mid \nu \mid$ at any $p \in M$.*

Remark 2.2. *If $r = 0$, each element φ from $G^r_F(M)$ is homeomorphism of manifold M. A Riemannian metric of the manifold M induces a Riemannian metric on each leaf L_α which defines a distance on it. In this case, φ is an isometry between the metric spaces L_α and $\varphi(L_\alpha)$. Then, according to the known theorem, φ is a diffeomorphism of L_α onto $\varphi(L_\alpha)$ for each leaf L_α and it's differential preserves the length of each tangent vector $\nu \in T_pF$, i. e. $\mid d\varphi_p(\nu) \mid = \mid \nu \mid$ at any $p \in M$ [3, page 56]. But as shown by a simple example, from differentiability of a mapping on each leaf can not imply it's differentiability on the entire manifold M.*

Example. Let $M = R^2(x, y)$ be the Euclidean plane with the Cartesian coordinates (x, y). Leaves L_α of foliation F are given by the equations $y = \alpha = const$. Then the plan homeomorphism $\varphi : R^2 \to R^2$ determined by the formula

$$\varphi(x, y) = (x + y, y^{\frac{1}{3}})$$

is an isometry of the foliation F, but is not a diffeomorphism of the plane.

The set $Diff^r(M)$ of all diffeomorphisms of a manifold M onto itself is a group with the operations of composition and taking the inverse. The set $G^r_F(M)$ is a subgroup of the group $Diff^r(M)$.

Let $\{K_\lambda\}$ be a family of all compact sets where each K_λ is a subset of some leaf of foliation F and let $\{U_\beta\}$ be a family of all open sets on M. We consider, for each pair K_λ and U_β, the set of all mappings $f \in G^r_F(M)$, such that $f(K_\lambda) \subset U_\beta$. This set of mappings is denoted by $[K_\lambda, U_\beta] = \{f : M \to M | f(K_\lambda) \subset U_\beta\}$.

It isn't difficult to show that all finite intersections of sets of the form $[K_\lambda, U_\beta]$ forms a base for some topology. This topology we will call the foliated compact open topology or, briefly $F-$ compact open topology.

The following proposition can be proved using a standard argument.

Proposition 2.1. *The set $G^r_F(M)$ with the $F-$ compact open topology is a Hausdorff space.*

The following theorem shows some property of group $G_F^r(M)$ with $F-$ compact open topology.

Theorem 2.1. *Let M be a complete smooth $n-$ dimensional manifold with the smooth $k-$ dimensional foliation F, $f_m \in G_F^r(M)$, $\quad r \geq 0, \quad m = 1, 2, 3, ...$ Suppose, that for each leaf L_α there exists a point $o_\alpha \in L_\alpha$ such that the sequence $f_m(o_\alpha)$ is convergent. Then there exists a subsequence f_{m_l} of the sequence f_m which converges in a $F-$ compact open topology.*

We denote by $d(x, y)$ the distance between the points x and y, determined by some complete Riemannian metric. It is known, a smooth manifold M possesses a complete Riemannian metric [2, page 186].

Theorem 2.2. *Let M be a smooth complete Riemannian manifold of dimension n with a smooth foliation F of dimension k, where $0 < k < n$. Then*

1) Each leaf with the induced Riemannian metric is a complete Riemannian manifold.

2) Let $\gamma_m : R^1 \to L_m$ be a sequence of geodesics (determined by the induced Riemannian metrics) on leaves L_m. If $\gamma_m(s_0) \to p$ for $m \to \infty$ for some $s_0 \in R^1$, then there exists a subsequence γ_{m_l} of the sequence γ_m which pointwise converges to some geodesic $\gamma : R^1 \to L(p)$ of leaf $L(p)$, passing through the point p at $s = s_0$.

3. Proof of the theorems

Proof. Theorem 2.2. 1) It is known, that for a connected manifold M following conditions are equivalent [4, page 167]:

a) M is a complete Riemannian manifold;

b) M is a complete metric space with the distance which is defined by a Riemannian metric.

Let L_α be some leaf of the foliation F, $\gamma : (a; b) \to L_\alpha$ the geodesic line on L_α, determined by the induced Riemannian metric and is parameterized by length of the arc. Let us show that, if $b < \infty$ (or $\quad a > -\infty$) then there exists $p = \lim_{s \to b} \gamma(s)$ (or $\quad \lim_{s \to a} \gamma(s)$) and $p \in L_\alpha$. Let $s_1 < s_2 < s_3 < \cdots < s_l < s_{l+1} < \ldots,\ s_l \to b$ for $l \to \infty$, and $\varepsilon > 0$ be some small number. Then there exists a number N such that $d(\gamma(s_i), \gamma(s_j))) \leq |s_i - s_j| < \varepsilon$ for $i, j \geq N$, where d is the distance on M. Since, the sequence $\{\gamma(s_l\}$ is a Cauchy sequence. By virtue of completeness of M, the sequence $\{\gamma(s_l\}$ is convergent in M to the some point p. Now we will show that $p \in L_\alpha$.

By the definition of a foliation for each point $p \in M$ there exists a neighborhood U of the point p and a local system of coordinates

$$(x^1, x^2, \ldots, x^k, y^{k+1}, y^{k+2}, \ldots, y^n)$$

on U such, that the set $\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^k}\}$ is basis for smooth sections of $TF|_U$ (restriction TF on U). Such a neighborhood is called a foliated neighborhood of the point p [9, page 122].

Let U be a foliated neighborhood of the point p with local coordinates

$$(x^1, x^2, \ldots, x^k, y^{k+1}, y^{k+2}, \ldots, y^n).$$

Then connected components of the intersection $U \bigcap L_\beta$ for any leaf L_β, are given by the equations: $y^{k+1} = const, y^{k+2} = const, \ldots, y^n = const$.

If L_0 is a connected component of $U \bigcap L_\alpha$, which contains points $\{\gamma(s_l)\}$ for enough large integer number l, then it is easy to check that L_0 contains the point p. Hence $p \in L_\alpha$. From here follows that $\gamma(s)$ is defined by for all $s \in (-\infty; +\infty)$.

2) Let $\pi : TM \to TF$ be orthogonal projection, $V(M), V(F), V(H)$ be the set of smooth sections of bundles TM, TF, HF respectively. We put $\widetilde{\nabla}_X Y = \pi(\nabla_X Y)$ for vector fields $X \in V(M), \quad Y \in V(F)$, where ∇ is a Levi-Civita connection, determined by the Riemannian metric g on M. It is known, that $\widetilde{\nabla}_X Y$ is a connection on TF, and it's restriction to each leaf L_α

coincides with connection on L_α, determined by the induced Riemannian metric on L_α from M ([5, page 20], [10, page 59]). Therefore, the smooth parametric curve $\mu : (a,b) \to M$ lying on a leaf L_α of the foliation F, is geodesic on L_α (determined by the induced Riemannian metric) if and only if

$$\widetilde{\nabla}_{\dot{\mu}}\dot{\mu} = 0. \tag{1}$$

If μ lies in the foliated neighborhood U, it's equations have the form:

$$\begin{cases} x^i = x^i(s) \\ y^\alpha = const, \end{cases}$$

where $1 \le i \le k, \quad k+1 \le \alpha \le n$. So, for ∇ we have

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^l_{i,j} \frac{\partial}{\partial x^l} + \Gamma^\alpha_{i,j} \frac{\partial}{\partial y^\alpha}, \tag{2}$$

hence,

$$\widetilde{\nabla}_{\frac{\partial}{\partial x^i}} = \Gamma^l_{i,j} \frac{\partial}{\partial x^l}, \tag{3}$$

where $1 \le i,j,l \le k, \quad k+1 \le \alpha \le n, \quad \Gamma^\beta_{i,j}$ are Christoffel symbols. From here using properties of the operator $\widetilde{\nabla}$ it follows, that equation (1) is equivalent to the following system of the differential equations of the 2-nd order:

$$\frac{d^2x^i}{ds} + \Gamma^i_{l,j} \frac{dx^l}{ds} \frac{dx^j}{ds} = 0. \tag{4}$$

If we put $u^i = \frac{dx^i}{ds}$, then it is possible to write this system as:

$$\begin{cases} \frac{dx^i}{ds} = u^i \\ \frac{du^i}{ds} = -\Gamma^i_{l,j} u^l u^j. \end{cases} \tag{5}$$

With no loss of generality, we can assume those geodesics $\gamma_m : R^1 \to L_m$ are parameterized by length of the arc and $s_0 = 0$. Now we show that there exists a subsequence of the sequence of tangential vectors $\dot{\gamma}_m(0)$, which converges. Since $|\dot{\gamma}_m(0)| = 1$ for each m, we follow equality

$$\sum_{i,j=1}^{k} g_{ij}(p_m) u^i_m u^j_m = 1 \tag{6}$$

for enough large integer number m, where $p_m = \gamma_m(0)$, g_{ij} are coefficients of the Riemannian metric g, $\{u^i_m\}$ are the first k coordinates of vector $\dot{\gamma}_m(0)$ on the foliated neighborhood U of the point p. Equality (6) is meaningful for $\gamma_m(0) \in U$. Since $p_m \to p$ at $m \to \infty$, for each $\varepsilon > 0$ there exists m_0 such that, $|g_{ij}(p_m) - g_{ij}(p)| < \varepsilon$, at $m \ge m_0$. From here we have

$$\sum_{i,j=1}^{k} (g_{ij}(p) - \varepsilon)|u^i_m||u^j_m| \le \sum_{i,j=1}^{k} g_{ij}(p_m)|u^i_m||u^j_m|.$$

By virtue of that a matrix $\{g_{ij}(p_m)\}$ is positively determined, each equation

$$\sum_{i,j=1}^{k} g_{ij}(p_m) u^i u^j = 1$$

the center of symmetry at the origin of coordinates determines ellipsoid.

By choosing new coordinate systems we can assume that

$$\sum_{i,j=1}^{k} g_{ij}(p_m)|u_m^i||u_m^j| = 1$$

and hence,

$$\sum_{i,j=1}^{k} (g_{ij}(p) - \varepsilon)|u_m^i||u_m^j| \leq 1. \tag{7}$$

For a sufficiently small number $\varepsilon > 0$, the matrix $\{(g_{ij}(p) - \varepsilon)\}$ is positively determined too. From here we have that points $u_m = \{u_m^1, u_m^2, \dots, u_m^k\}$ belong to a compact set. Therefore there exists a converging subsequence $u_{m_l}^i$ of the sequence u_m^i,. We denote by u_0^i limit of the $u_{m_l}^i$. Then for vector $v = (u_0^1, u_0^2, \dots, u_0^k)$ has places

$$\sum_{i,j=1}^{k} g_{ij}(p)|u_0^i||u_0^j| = 1.$$

Let us consider a geodesic γ on the leaf $L(p)$ passing through the point p at $s = 0$ in a direction of vector v. This curve satisfies the equation (5). Let $K_0 \subset R^1$ be compact containing $s_0 = 0$ such, that $\gamma(K_0) \subset U$. Then coordinate functions of the curve $\gamma : K_0 \to L(p)$ satisfy to the system of the differential equations (5) with the initial conditions: $x^i(0) = p^i, \quad u^i(0) = v^i$, where $i = 1, 2, \dots, k, \quad p = (p^1, p^2, \dots, p^n), \quad v = (u_0^1, u_0^2, \dots, u_0^k)$.

As $\gamma_{m_l}(0) \to p, \quad \dot{\gamma}_{m_l}(0) \to v$ for $m \to \infty$ under the theorem of continuous dependence of the solution of the differential equations from the initial dates the sequence γ_m is convergent to γ uniformly on compact $K_0 \subset R^1$. Further for every compact $K \subset R^1$, containing K_0, covering $\gamma(K)$ with the foliated neighborhoods, we shall receive, that γ_{m_l} is convergent to γ uniformly on compact K. The Theorem 2.2 has been proved. □

Lemma 3.1. *Assume that a sequence $\{f_m\} \in G_F^r(M)$ converges pointwise on a set $A \subset L_\alpha$ where $L_\alpha-$ some leaf of the foliation F. Then $\{f_m\}$ also converges pointwise on $\overline{A}$ (where $\overline{A}-$ the closure of the set A in L_α).*

Proof. Let $p \in \overline{A}$, $\varepsilon > 0$. At first we select a point $p_1 \in A$, that $d_\alpha(p, p_1) < \frac{\varepsilon}{3}$ and an integer N such that $d(f_l(p_1), f_m(p_1)) < \frac{\varepsilon}{3}$ for $l, m \geq N$ (where $d_\alpha(p, p_1)$ is the distance between points p and p_1 on the leaf L_α). Then

$$d(f_l(p), f_m(p)) \leq d(f_l(p), f_l(p_1)) + d(f_l(p_1), f_m(p_1)) + d(f_m(p_1), f_m(p)) < \varepsilon,$$

for all m, n at $m, n \geq N$. Hence, $\{f_m(p)\}$ is fundamental and from completeness of M follows that $\{f_m(p)\}$ is convergent. □

Lemma 3.2. *Let A be a set of points on a leaf L_α such that for each point $p \in A$ there exists a converging subsequence $f_{m_l}(p)$ of the sequence $f_m(p)$. If the set A is nonempty set, then $A = L_\alpha$.*

Proof. Let $p \in L_\alpha, p^* \in A, r = d_\alpha(p, p^*)$, where $d_\alpha(p, p^*)$ is the distance between points p and p^* on the leaf L_α. Assume that $\{f_{m_l}\}$ is a subsequence that $\{f_{m_l}(p^*)\}$ is convergent. Since f_{m_l} an isometries of foliation, the distance $d_\alpha(p, p^*)$ between points p and p^* on the leaf L_α is preserved and holds equality $d_{f_l(\alpha)}(f_{m_l}(p), f_{m_l}(p^*)) = d_\alpha(p, p^*)$ where $d_{f_l(\alpha)}$ is the distance on the leaf $f_{m_l}(L_\alpha)$.

Let $q^* = \lim_{l \to \infty} (f_{m_l}(p^*))$. Then

$$d(q^*, f_{m_l}(p)) \leq d(q^*, f_{m_l}(p^*)) + d(f_{m_l}(p^*), f_{m_l}(p)) \leq \varepsilon + r.$$

Hence, from completeness of M follows that the set $\{f_{m_l}(p)\}$ has compact closure. It follows that $p \in A$. □

Proof. Theorem 2.1. Let $p \in L_\alpha$ for a some leaf L_α. Under conditions of the theorem there exists a point $o_\alpha \in L_\alpha$, such that $f_m(o_\alpha)$ is convergent. From Lemma 3.2 for each point $p \in L_\alpha$ the sequence $\{f_m(p)\}$ containing converging subsequence. Hence the sequence $\{f_m(p)\}$ containing subsequence which converges for all points $p \in M$.

Since Riemannian manifold M is a separable metric space it contains everywhere dense countable subset $A = \{p_i\}$. For each point p_i there is a converging subsequence $\{f^i_{m_l}(p_i)\}$ of the sequence $\{f_m(p)\}$. Using diagonal process, we can find a subsequence $\{f_{m_l}\}$ which converges at all points. Therefore for each leaf L_α there exists a point $o_\alpha \in L_\alpha$, such that sequence $\{f_{m_l}(o_\alpha)\}$ is convergent. By Lemma 3.2 this sequence is convergent at all points of L_α.

Now by putting $\varphi(p) = \lim\limits_{l\to\infty} f_{m_l}(p)$, we have the map $\varphi : M \to M$.

Let $p \in L_\alpha$ for some leaf L_α, $\gamma : [0, l] \to L_\alpha$ be a geodesic which realizes the distance $d_0 = d_\alpha(o_\alpha, p)$ on the leaf L_α,and is parameterized by length of the arc, $\gamma(0) = o_\alpha$, $\gamma(l) = p$. If we consider $\gamma_l = f_{m_l}(\gamma)$, then they are geodesic lines on the leaf $f_{m_l}(L_\alpha)$. From condition of theorem we have $\gamma_l(0) \to p_0$ at $l \to \infty$, where p_0 is a some point on M. From Theorem 2.2 follows that there exists subsequence of sequence $\{\gamma_l(s)\}$, which pointwise converges to some geodesic $\gamma_(0) : R^1 \to L(p_0)$ of the leaf $L(p_0)$, passing through the point p_0 at $s = 0$.

With no loss of generality, we can suppose that sequence $\{\gamma_l(s)\}$ is convergent to geodesic $\gamma_0(s)$ for each $s \in [0; l]$. Therefore follows that $\varphi(\gamma) = \gamma_0$, i.e. map φ is an isometry of L_α to $L(p_0)$.

Now we show that $f_{m_l} \to \varphi$ uniformly on each compact, lying on a leaf of foliation F. Let K be a compact set on a leaf L and $\varepsilon > 0$. As K is a compact set there exist finite points $p_1, p_2, \ldots, p_m$ on L such that each point $p \in K$ from some p_i has distance less than ε .

For each point p_i there is number N_i such that $d(f_{m_l}(p_i), \varphi(p_i)) < \frac{\varepsilon}{3}$ for any $m_l \geq N_i$. Besides for each point $p \in K$ there exists p_i such that $d_L(p, p_i) < \frac{\varepsilon}{3}$ where $d_L(p, p_i)$ distance between points p and p_i determined by induced Riemannian metric on L. Therefore follows that

$$d(f_{m_l}(p), \varphi(p)) \leq d(f_{m_l}(p), f_{m_l}(p_i)) + d(f_{m_l}(p_i), \varphi(p_i)) + d(\varphi(p_i), \varphi(p)) \leq$$
$$\leq d_{m_l}(f_{m_l}(p), f_{m_l}(p_i)) + d(f_{m_l}(p_i), \varphi(p_i)) + d_l(\varphi(p_i), \varphi(p)) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

at $m_l > N = \max\limits_{1\leq i\leq m} \{N_i\}$, where $1 \leq i \leq m$. From here follows that $f_{m_l} \to \varphi$ in $F-$ compact open topology. The Theorem 2.1 has been proved. □

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Abdigappar Narmanov, for a photograph and biography, see TWMS J. Pure Appl. Math., V.2, N.1, 2011, p.136